This was about the sum of our knowledge of these sheep when Mr. Sheldon set out in 1904 and 1905 to make a special study of the sheep question of the northern Rockies, and to trace out their geographical and physical relationships. Chapter XX. of this book gives a summary of the results of his two seasons' work, and is illustrated by a map in colors showing the known distribution of the white and black sheep of Canada and Alaska, their areas of intergradation and the phases characteristic of special districts. Facing the map are half-tone figures of nine stages of color variation, with explanatory text. The subject is thus graphically and clearly illustrated by the distribution map, the facing explanatory text and shaded figures. The area embraced extends from about latitude 55° to latitude 70°. In Alaska, from the Arctic coast south to latitude 60°, and in Yukon Territory and northeastward in the Mackenzie Mountains to about latitude 62° (generally speaking), the sheep are pure white, except in the Tanana Hills south of the Yukon River, where the white coat is varied with a few black hairs and slight indications of the color pattern of the *fannini* type; in British Columbia south of the Stikine River the sheep are uniformly black; but over an intervening region of from approximately six hundred and fifty miles north and south and about one hundred and fifty to two hundred miles east and west, "there is no area in which the color of the sheep is uniform."

Mr. Sheldon indicates on his map five areas (*a, b, c, d, e*) where the sheep are either pure white (*a*), or black (*e*), or are of intermediate or mixed shades (*b, c, d*); the *b* grade is nearly white, the *d* grade nearly black, *c* being the middle phase or the *fannini* type, which is intermediate geographically as well as in color.

The facts of intergradation are thus forcibly and clearly presented—an intergradation continuous and gradual from one extreme phase to the other through a vast expanse of country. The cause of this extensive and gradual merging of these two widely diverse color types of sheep is not so easily demonstrable. Has it resulted from interbreeding or is it due to environment? Mr. Sheldon favors the former hypothesis, but admits the possibility of its having been "produced by subtle and indeterminate changes of environment to a much greater extent than the facts seem to me [him] to warrant."

The large size of these animals and the striking color differences between the extreme phases that are thus shown to intergrade render this an impressive instance of intergradation, but parallel cases, though less striking, in other animals usually seem explainable satisfactorily, and in many instances beyond question, on the hypothesis of the action of diverse conditions of environment. But whatever conclusion may finally be reached as to the cause, great credit is due Mr. Sheldon for his contribution of facts through a successful reconnoissance of the almost inaccessible haunts of the sheep in the Northern Rockies where lay the key to the problem—an undertaking few would have the hardihood to project or the endurance and persistence to accomplish. Besides the facts of variation and range already outlined, his contribution to the life-history of these animals is of noteworthy importance, while the wide range of individual variation among members of the same herd, not only as regards coloration, but in respect to size, shape and curvature of the horns is noted in detail. He has also presented to the National Museum the large series of specimens of sheep obtained by him on his expeditions which go far to substantiate the facts of intergradation recorded and illustrated in his book, which may be read with equal interest by the naturalist, the big game hunter and the general reader.

J. A. Allen

Principia Mathematica. By Alfred North Whitehead, Sc.D., F.R.S., Fellow and Lecturer of Trinity College, Cambridge, and Bertrand Russell, M.A., F.R.S., Lecturer and late Fellow of Trinity College, Cambridge. Cambridge University Press. 1910. Vol. I., pp. xiii + 666.

Mathematicians, many philosophers, logicians and physicists, and a large number of other people are aware of the fact that mathematical activity, like the activity in numerous other fields of study and research, has been in large part for a century distinctively and increasingly critical. Every one has heard of a critical movement in mathematics and of certain mathematicians distinguished for their insistence upon precision and logical cogency. Under the influence of the critical spirit of the time mathematicians, having inherited the traditional belief that the human mind can know some propositions to be true, convinced that mathematics may not contain any false propositions, and nevertheless finding that numerous so-called mathematical propositions were certainly not true, began to reexamine the existing body of what was called mathematics with a view to purging it of the false and of thus putting an end to what, rightly viewed, was a kind of scientific scandal. Their aim was truth, not the whole truth, but nothing but truth. And the aim was consistent with the traditional faith which they inherited. They believed that there were such things as self-evident propositions, known as axioms. They believed that the traditional logic, come down from Aristotle, was an absolutely perfect machinery for ascertaining what was involved in the axioms. At this stage, therefore, they believed that, in order that a given branch of mathematics should contain truth and nothing but truth, it was sufficient to find the appropriate axioms and then, by the engine of deductive logic, to explicate their meaning or content. To be sure, one might have trouble to "find" the axioms and in the matter of explication one might be an imperfect engineer; but by trying hard enough all difficulties could be surmounted for the axioms existed and the engine was perfect. But mathematicians were destined not to remain long in this comfortable position. The critical demon is a restless and relentless demon; and, having brought them thus far, it soon drove them far beyond. It was discovered that an axiom of a given set could be replaced by its contradictory and that the consequences of the new set stood all the experiential tests of truth just as well as did the consequences of the old set, that is, perfectly. Thus belief in the self-evidence of axioms received a fatal blow. For why regard a proposition self-evident when its contradictory would work just as well? But if we do not know that our axioms are true, what about their consequences? Logic gives us these, but as to their being true or false, it is indifferent and silent.

Thus mathematics has acquired a certain modesty. The critical mathematician has abandoned the search for truth. He no longer flatters himself that his propositions are or can be known to him or to any other human being to be true; and he contents himself with aiming at the correct, or the consistent. The distinction is not annulled nor even blurred by the reflection that consistency contains immanently a kind of truth. He is not absolutely certain, but he believes profoundly that it is possible to find various sets of a few propositions each such that the propositions of each set are compatible, that the propositions of such a set imply other propositions, and that the latter can be deduced from the former with certainty. That is to say, he believes that there are systems of coherent or consistent propositions, and he regards it his business to discover such systems. Any such system is a branch of mathematics. Any branch contains two sets of ideas (as subject matter, a third set of ideas are used but are not part of the subject matter) and two sets of propositions (as subject matter, a third set being used without being part of the subject): a set of ideas that are adopted without definition and a set that are defined in terms of the others; a set of propositions adopted without proof and called assumptions or principles or postulates or axioms (but not as true or as self-evident) and a set deduced from the former. A system of postulates for a given branch of mathematics—a variety of systems may be found for a same branch—is often called the foundation of that branch. And that is what the layman should think when, as occasionally happens,

he meets an allusion to the foundation of the theory of the real variable, or to the foundation of Euclidean geometry or of projective geometry or of *Mengenlehre* or of some other branch of mathematics. The founding, in the sense indicated, of various distinct branches of mathematics is one of the great outcomes of a century of critical activity in the science. It has engaged and still engages the best efforts of men of genius and men of talent. Such activity is commonly described as fundamental. It is very important, but fundamental in a strict sense it is not. For one no sooner examines the foundations that have been found for various mathematical branches and thereby as well as otherwise gains a deep conviction that these branches are constituents of something different from any one of them and different from the mere sum or collection of all of them than the question supervenes whether it may not be possible to discover a foundation for mathematics itself such that the above-indicated branch foundations would be seen to be, not fundamental to the science itself, but a genuine part of the superstructure. That question and the attempt to answer it are fundamental strictly. The question was forced upon mathematicians not only by developments within the traditional field of mathematics, but also independently from developments in a field long regarded as alien to mathematics, namely, the field of symbolic logic. The emancipation of logic from the yoke of Aristotle very much resembles the emancipation of geometry from the bondage of Euclid; and, by its subsequent growth and diversification, logic, less abundantly perhaps but not less certainly than geometry, has illustrated the blessings of freedom. When modern logic began to learn from such a man as Leibniz (who with the most magnificent expectations devoted much of his life to researches in the subject) the immense advantage of the systematic use of symbols, it soon appeared that logic could state many of its propositions in symbolic form, that it could prove some of these, and that the demonstration could be conducted and expressed in the language of symbols. Evidently such a logic looked like mathematics and acted like it. Why not call it mathematics? Evidently it differed from mathematics in neither spirit nor form. If it differed at all, it was in respect of content. But where was the decree that the content of mathematics should be restricted to this or that, as number or space? No one could find it. If traditional mathematics could state and prove propositions about number and space, about relations of numbers and of space configurations, about classes of numbers and of geometric entities, modern logic began to prove propositions about propositions, relations and classes, regardless of whether such propositions, relations and classes have to do with number and space or any other specific kind of subject. At the same time what was admittedly mathematics was by virtue of its own inner developments transcending its traditional limitations to number and space. The situation was unmistakable: traditional mathematics began to look like a genuine part of logic and no longer like a separate something to which another thing called logic applied. And so modern logicians by their own researches were forced to ask a question, which under a thin disguise is essentially the same as that propounded by the bolder ones among the critical mathematicians, namely, is it not possible to discover for logic a foundation that will at the same time serve as a foundation for mathematics as a whole and thus render unnecessary (and strictly impossible) separate foundations for separate mathematical branches?

It is to answer that great question that Messrs. Whitehead and Russell have written "Principia Mathematica"—a work consisting of three large volumes, the first being in hand, the second and third soon to appear—and the answer is affirmative. The thesis is: it is possible to discover a small number of ideas (to be called primitive ideas) such that all the other ideas in logic (including mathematics) shall be definable in terms of them, and a small number of propositions (to be called primitive propositions) such that all other propositions in logic (including mathematics) can be demonstrated by means of

them. Of course, not all ideas can be defined —some must be assumed as a working stock— and those called primitive are so called merely because they are taken without definition; similarly for propositions, not all can be proved, and those called primitive are so called because they are assumed. It is not contended by the authors (as it was by Leibniz) that there exist ideas and propositions that are absolutely primitive in a metaphysical sense or in the nature of things; nor do they contend that but one sufficient set of primitives (in their sense of the term) can be discovered. In view of the immeasurable wealth of ideas and propositions that enter logic and mathematics, the authors' thesis is very imposing; and their work borrows some of its impressiveness from the magnificence of the undertaking. It is important to observe that the thesis is not a thesis of logic or of mathematics, but is a thesis about logic and mathematics. It can not be proved syllogistically; the only available method is that by which one proves that one can jump through a hoop, namely, by actually jumping through it. If the thesis be true, the only way to establish it as such is to produce the required primitives and then to show their adequacy by actually erecting upon them as a basis the superstructure of logic (and mathematics) to such a point of development that any competent judge of such architecture will say: "Enough! I am convinced. You have proved your thesis by actually performing the deed that the thesis asserts to be possible."

And such is the method the authors have employed. The labor involved—or shall we call it austere and exalted play?—was immense. They had predecessors, including themselves. Among their earlier works Russell's "Principles of Mathematics" and Whitehead's "Universal Algebra" are known to many. The related works of their predecessors and contemporaries, modern critical mathematicians and modern logicians, Weierstrass, Cantor, Boole, Peano, Schröder, Peirce and many others, including their own former selves, had to be digested, assimilated and transcended. All this was done, in the course of more than a score of years; and the work before us is a noble monument to the authors' persistence, energy, acumen and idealism. A people capable of such a work is neither crawling on its belly nor completely saturated with commercialism nor wholly philistine. There are preliminary explanations in ordinary language and summaries and other explanations are given in ordinary language here and there throughout the book, but the work proper is all in symbolic form. Theoretically the use of symbols is not necessary. A sufficiently powerful god could have dispensed with them, and, unless he were a divine spendthrift, he would have done so, except perhaps for the reason that whatever is feasible should be done at least once in order to complete the possible history of the world. But whilst the employment of symbols is theoretically dispensable, it is, for man, practically indispensable. Many of the results in the work before us could not have been found without the help of symbols, and even if they could have been thus found, their expression in ordinary speech, besides being often unintelligible, owing to complexity and involution, would have required at least fifteen large volumes instead of three. Fortunately the symbology is both interesting and fairly easy to master. The difficulty inheres in the subject itself.

The initial chapter, devoted to preliminary explanations that any one capable of nice thinking may read with pleasure and profit, is followed by a chapter of 30 pages dealing with "the theory of logical types." Mr. Russell has dealt with the same matter in volume 30 of the *American Journal of Mathematics* (1908). One may or may not judge the theory to be sound or adequate or necessary and yet not fail to find in the chapter setting it forth both an excellent example of analytic and constructive thinking and a worthy model of exposition. The theory, which, however, is recommended by other considerations, originated in a desire to exclude from logic automatically by means of its principles what are called illegitimate totalities and therewith a subtle variety of contradiction and vicious circle fallacy that, owing their presence to the

non-exclusion of such totalities, have always infected logic and justified skepticism as to the ultimate soundness of all discourse, however seemingly rigorous. (Such theoretic skepticism may persist anyhow, on other grounds.) Perhaps the most obvious example of an illegitimate totality is the so-called class of all classes. Its illegitimacy may be shown as follows. If A is a class (say that of men) and E is a member of it, we say, E is an A. Now let W be the class of all classes such that no one of them is a member of itself. Then, whatever class x may be, to say that x is a W is equivalent to saying that x is not an x, and hence to say that W is a W is equivalent to saying that W is not a W! Such illegitimate totalities (and the fallacies they breed) are in general exceedingly sly, insinuating themselves under an endless variety of most specious disguises, and that, not only in the theory of classes but also in connection with every species of logical subject-matter, as propositions, relations and propositional functions. As the propositional function—any expression containing a real (as distinguished from an apparent) variable and yielding either non-sense or else a proposition whenever the variable is replaced by a constant term—is the basis of our authors' work, their theory of logical types is fundamentally a theory of types of propositional functions. It can not be set forth here nor in fewer pages than the authors have devoted to it. Suffice it to say that the theory presents propositional functions as constituting a summitless hierarchy of types such that the functions of a given type make up a legitimate totality; and that, in the light of the theory, truth and falsehood present themselves each in the form of a systematic ambiguity, the quality of being true (or false) admitting of distinctions in respect of order, level above level, without a summit. When Epimenides, the Cretan, says that all statements of Cretans are false, and you reply that then his statement is false, the significance of "false" here presents two orders or levels; and logic must by its machinery automatically prevent the possibility of confusing them.

Next follows a chapter of 20 pages, which all philosophers, logicians and grammarians ought to study, a chapter treating of Incomplete Symbols wherein by ingenious analysis it is shown that the ubiquitous expressions of the form "the so and so" (the "the" being singular, as "the author of Waverley," "the sine of *a*," "the Athenian who drank hemlock," etc.) do not of themselves denote anything, though they have contextual significance essential to discourse, essential in particular to the significance of identity, which, in the world of discourse, takes the form of "*a* is the so and so" and not the form of the triviality, *a* is *a*.

After the introduction of 88 pages, we reach the work proper (so far as it is contained in the present volume), namely, Part I.: Mathematical Logic. Here enunciation of primitives is followed by series after series of theorems and demonstrations, marching through 578 pages, all matter being clad in symbolic garb, except that the continuity is interrupted here and there by summaries and explanations in ordinary language. Logic it is called and logic it is, the logic of propositions and functions and classes and relations, by far the greatest (not merely the biggest) logic that our planet has produced, so much that is new in matter and in manner; but it is also mathematics, a prolegomena to the science, yet itself mathematics in the most genuine sense, differing from other parts of the science only in the respects that it surpasses these in fundamentality, generality and precision, and lacks traditionality. Few will read it, but all will feel its effect, for behind it is the urgence and push of a magnificent past: two thousand five hundred years of record and yet longer tradition of human endeavor to think aright.

C. J. Keyser

Columbia University

A LETTER OF LAMARCK

Letters of Lamarck are not often found. M. Landrieux, who has recently published a life of Lamarck, states that "one can count the number of his letters which have come